

FINDING GEODESICS IN A TRIANGULATED 2-SPHERE

ABSTRACT. Let S be a triangulated 2-sphere with fixed triangulation T . We apply the methods of thin position from knot theory to obtain a simple version of the three geodesics theorem for the 2-sphere [5]. In general these three geodesics may be unstable, corresponding, for example, to the three equators of an ellipsoid. Using a piece-wise linear approach, we show that we can usually find at least three stable geodesics.

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1. INTRODUCTION

Let S be a triangulated 2-sphere with fixed triangulation T . We assume T is a simplicial complex, see [3]. In particular we may assume that the 1-skeleton of T contains no loops or multiple edges. We apply the methods of thin position from knot theory to obtain a simple version of the three geodesics theorem for the 2-sphere [5]. Using this piece-wise linear approach we can go further, and strengthen the result to find at least three stable (PL) geodesics, unless the triangulation is the tetrahedral triangulation or the “double tetrahedral” triangulation.

1.1. Outline of the paper. In section 2, we define stable and unstable geodesics, and thin position for a triangulation of the 2-sphere. We prove the basic result that a thin triangulation naturally yields geodesics corresponding to stable and unstable geodesics. In section 3 we define bridge position for a triangulation, analogous to bridge position for a knot in the 3-sphere. We use a result of H. Whitney on the existence of Hamiltonian cycles to examine the relation between thin position and bridge position for a triangulation, and conclude that thin position is the same as bridge position only in the case of the tetrahedral triangulation. In sections 4 and 5 we pursue this idea, and use it to obtain a relatively simple version of the three geodesics theorem, in which the three geodesics are allowed to be either stable or unstable. Finally in section 6 we refine our analysis of a thin triangulation to obtain, with two exceptions, the existence of three stable geodesics for a triangulated 2-sphere.

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2. WIDTH OF A TRIANGULATION

Let S be a triangulated 2-sphere with fixed triangulation T .

1. Definition. Let $P = e_1, e_2, \dots, e_k$ be an imbedded cycle in the edges of T . Let T_j be a triangle in T that intersects P in exactly one or exactly two (necessarily adjacent) edges in P . A *local move on P* replaces the one or two edges of T_j with two or one edges of T_j , yielding another imbedded cycle Q with either one more or one fewer edges than P . Call the first kind of move a *shortening* of P ; the second a *lengthening*.

2. Definition. Let $P = e_1, e_2, \dots, e_k$ be an imbedded cycle in the edges of T . P is a *stable geodesic* if it allows no shortening moves and P is not the boundary of a triangle.

3. Definition. Let $P = e_1, e_2, \dots, e_k$ be an imbedded cycle in the edges of T . P divides S into two disks, D_1 and D_2 . Suppose P has two shortening moves, one in D_1 across T_1 and one in D_2 across T_2 . Suppose further that for every such pair, T_1 and T_2 intersect in an edge e_3 contained in P . Notice that this intersection prevents P from shortening to both sides simultaneously. We call such a P an *unstable geodesic* (see Figure 1).

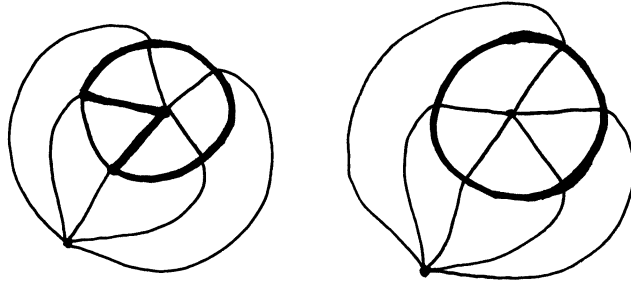


FIGURE 1. unstable vs. stable geodesic

4. Definition. Since triangulations of the 2-sphere are shellable [1], we can choose an order $O(T)$ for the triangles of T , T_1, T_2, \dots, T_n so that

$$I_k = T_1 \cup T_2 \cup \dots \cup T_k$$

is homeomorphic to a disk for $k < n$. Call such an order *good*. We assume for the remainder of this paper that a specified order for a given T is good.

5. Definition. Let $O(T)$ be an ordering of T . Call the number of vertices of T in the boundary of I_k the *length of $\partial(I_k)$* , and denote it $|\partial(I_k)|$. Notice that in a good ordering, the addition of each successive triangle either increases the length of the boundary of the disk by exactly one or reduces it by exactly one. A *local maximum* of the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$ is a value $|\partial I_j|$ such that

$$|\partial I_{j-1}| < |\partial I_j| > |\partial I_{j+1}|,$$

$j = 1, \dots, n-1$. A *local minimum* of the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$ is a value $|\partial I_j|$ such that

$$|\partial I_{j-1}| > |\partial I_j| < |\partial I_{j+1}|,$$

$j = 1, \dots, n-1$. We say T with order $O(T)$ is in *bridge position* if the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$ has a single local maximum and no local minima (see Figure 2).

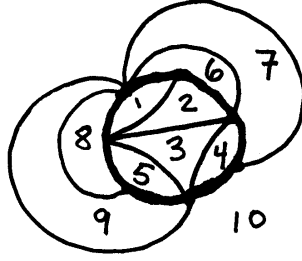


FIGURE 2. bridge position; local max at ∂I_5

The *width of $O(T)$* , $w_O(T)$, is the list of local maxima of $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$, lexicographically ordered. The *width of (T)* , $w(T)$, is the minimum over all such lists, lexicographically ordered.

We say that T with order $O(T)$ is in *thin position* if $O(T)$ realizes the width of T . A local maximum (minimum) *corresponds to* the cycle which is the boundary of the disk ∂I_j .

6. Theorem. *If T with order $O(T)$ is in thin position, then*

- (1) *the cycles corresponding to the local maxima are unstable geodesics*
- (2) *the cycles corresponding to the local minima are stable geodesics.*

Proof

We start with two technical claims. We introduce the dual graph of T , Γ_T , which is useful when analyzing how $|\partial(I_k)|$ changes when a triangle is added or removed.

7. Claim. *Let D be a triangulated disk with a (good) ordering T_1, T_2, \dots, T_m such that $(|\partial I_1| < |\partial I_2| < \dots < |\partial I_m|)$. Let Γ_T be the dual graph in D . Then Γ_T is a tree.*

Proof

We can build Γ_T following the order on T . Since $|\partial(I_k)|$ is strictly increasing as k increases from 1 to m , as Γ_T is built each new vertex must have degree one, hence Γ_T is a tree.

8. Claim. *Let D be a triangulated disk with a (good) ordering $O(T)$, T_1, T_2, \dots, T_m . Suppose there is a shortening move for ∂D across T_i . Then the ordering $O_*(T)$ given by $T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_m, T_i$ is also a good ordering, and $w_{O_*}(T) \leq w_O(T)$.*

Proof

Note that since there is a shortening move for ∂D across T_i , T_i corresponds to a valence one vertex in Γ_T . Thus the homeomorphism type of $T_1 \cup T_2 \cup \dots \cup T_{i-1} \cup T_{i+1} \cup \dots \cup T_j$ is the same as that of I_j , hence the ordering $T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_m, T_i$ is still good. Each local maximum in the list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_m|)$ is either unchanged or reduced by one when the addition of T_i is delayed to the last step. Call this the *re-ordering principle* (see Figure 3).



FIGURE 3. re-ordering shaded triangle

Note that the re-ordering principle applies more generally. Suppose T_1, T_2, \dots, T_m is a triangulation of a planar region P and suppose there is a shortening move across T_i for a boundary curve C of P . We can define the width of this ordering for P as before. By the argument above, each local maximum in the list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_m|)$ is either unchanged or reduced by one when the addition of T_i is delayed, and the homeomorphism type of the region at each stage is unchanged if the addition of T_i is delayed, so that we may assume $i = m$.

Suppose T with order $O(T)$ is in thin position, and suppose C is a cycle corresponding to a local maximum $|\partial I_j|$.

Since C corresponds to a local maximum, C has shortening moves to both sides. By the re-ordering principle, we can assume that the triangles corresponding to these shortening moves are T_j and T_{j+1} .

If T_j and T_{j+1} are disjoint or share a vertex, one can check that the new ordering obtained by interchanging T_j and T_{j+1} ,

$$O'(T) : T_1, \dots, T_{j-1}, T_{j+1}, T_j, T_{j+2}, \dots, T_n$$

is also good, but $w_{O'}(T) < w_O(T)$.

This contradicts the hypothesis that $O(T)$ is thin, hence C is an unstable geodesic, proving part 1 of the theorem. To conclude the proof, we consider what happens between two local minima:

9. Definition. Suppose T with order $O(T)$ is in thin position. Suppose ∂I_i and ∂I_k are cycles in T corresponding to local minima of $O(T)$ such that the ordered list $(|\partial I_i|, |\partial I_{i+1}|, \dots, |\partial I_k|)$ has a single local maximum. Then ∂I_i and ∂I_k are *adjacent* minima in T . If the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_i|)$ has a single local maximum not at $|\partial I_i|$ we say that ∂I_i is adjacent to the empty geodesic (see Figure 4).

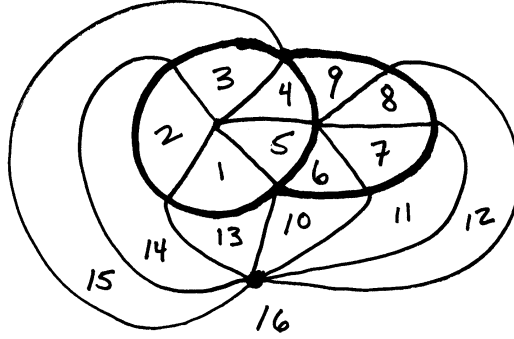


FIGURE 4. ∂I_5 and ∂I_9 are adjacent local minima

Now suppose to the contrary that some cycle corresponding to a minimum of $O(T)$ at $|\partial I_i|$ has a shortening move. Then either there exists one such move corresponding to a triangle lying between adjacent minima, or ∂I_i is adjacent to the empty geodesic and there is a shortening move for ∂I_i in the disk I_i .

Suppose $|\partial I_i|$ and $|\partial I_k|$ are adjacent minima in T , corresponding to cycles C_i and C_k . Assume the single maxima between them occurs at C_j . Suppose C_i has a shortening move across T_h and T_h is contained in the region between C_i and C_k . By the re-ordering principle, we can re-order the triangles between C_i and C_k , without increasing the width, so that $h = i + 1$. Since the addition of each triangle exactly increases or exactly decreases the length of the disk boundary by 1, the number of triangles between C_i and C_k is exactly $(|C_j| - |C_i|) + (|C_j| - |C_k|)$. When we re-order the triangles so that $h = i + 1$, the maximum length achieved is at least one smaller than $|C_j|$, hence the overall width is smaller than $O(T)$. This contradicts thinness of O , hence C_i cannot have a shortening move across T_h with T_h contained in the region between C_i and C_k .

Suppose ∂I_i is adjacent to the empty geodesics and there is a shortening move for ∂I_i in the disk I_i . By the reordering principle we can reorder the triangulation so that the triangle associated to the shortening move is T_i . However the width of this reordering is lower than that of the original ordering, contradicting thinness.

Hence no cycle corresponding to a minimum has any shortening move, hence all such cycles are stable geodesics, as required.

3. WIDTH AND HAMILTONIAN CYCLES

A theorem of H. Whitney gives sufficient conditions for T to contain a Hamiltonian cycle. We examine the relation between the existence of a Hamiltonian cycle and bridge position for T . Recall that T with order $O(T)$ is in bridge position if the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$ has a single local maximum and no local minima.

10. Theorem. [6] *If every cycle of length three in T is the boundary of a triangle in T , then T has a Hamiltonian cycle.*

11. Theorem. *T has a Hamiltonian cycle if and only if T has an order $O(T)$ so that T with order $O(T)$ is in bridge position.*

Proof

Suppose T has a Hamiltonian cycle. Let D_1 and D_2 be the two disks (thought of in S^2) defined by the Hamiltonian cycle. Let Γ_i be the graph dual to T in D_i . Since all the vertices of T lie on ∂D_i , Γ_i is a tree. Construct the desired (good) order $O(T)$ by constructing Γ_1 from a root to the leaves, and then reversing the process for Γ_2 .

Conversely suppose the ordering $O(T)$ on T has a unique local maximum and no local minima. Suppose $|\partial I_j|$ is the unique local maximum for $O(T)$. Then ∂I_j is a Hamiltonian cycle for T .

12. Corollary. *If every cycle of length three in T is the boundary of a triangle in T then T has an order $O(T)$ so that T with order $O(T)$ is in bridge position.*

4. WHEN THIN EQUALS BRIDGE

13. Theorem. *Let T be a triangulation of the 2-sphere. Suppose T with order $O(T)$ is in both thin position and bridge position. Then T is the tetrahedral triangulation of S^2 .*

Proof

Suppose T with order $O(T)$ is in both thin position and bridge position. Let I_k be the disk such that $|\partial I_k|$ realizes the single local maximum of $O(T)$. Let $J_k = S^2 - I_k$.

By Theorem 11, ∂I_k is a Hamiltonian cycle in the 1-skeleton of T . By Theorem 6, ∂I_k is an unstable geodesic, so cannot have disjoint, or 1-point intersecting, shortening moves in I_k and J_k .

Each of I_k and J_k have at least two distinct outermost arcs, else a single arc which is outermost to both sides. Each outermost arc corresponds to a shortening move, since there are no vertices in the interior of I_k or J_k . Let a_1 be an outermost arc of I_k , b be an outermost arc of J_k . Then the endpoints of a_1 and b must be nested on $\partial(I_k) = \partial(J_k)$, else there will be shortening moves corresponding to a and b which are disjoint (see Figure 5).

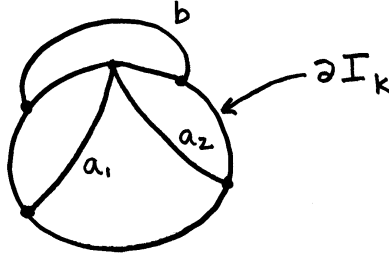


FIGURE 5. outermost arcs

This nesting must hold for all possible pairs of outermost arcs in I_k and J_k . Suppose a_2 is a distinct outermost arc of I_k . Since it must also have nested endpoints with b , it must share exactly one endpoint with a_1 . An additional outermost arc of J_k will have to nest both with a_1 and with a_2 , forcing it to coincide with b . Hence J_k has exactly

one outermost arc, hence I_k has exactly one outermost arc, and the theorem follows.

Note that the tetrahedral triangulation has three length four unstable geodesics, similar to the smooth case of an ellipsoid with three distinct radii.

5. WHEN THIN DOES NOT EQUAL BRIDGE

14. Theorem. *Let T be a triangulation of the 2-sphere. Suppose T with order $O(T)$ is in thin position but not bridge position. Then T has at least three distinct geodesics.*

Proof

Suppose T with order $O(T)$ is in thin position but not bridge position. Then the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_{n-1}|)$ has at least two local maxima, say at $|\partial I_i|$ and $|\partial I_k|$, and at least one local minima, say at $|\partial I_j|$. Hence T has at least two unstable geodesics, ∂I_i and ∂I_k and one stable geodesic, ∂I_j . While distinct, they may overlap in paths.

15. Corollary. *Let T be a triangulation of the 2-sphere. Then T has at least three distinct geodesics.*

Proof

T is either the tetrahedral triangulation or there exists $O(T)$ such that T with order $O(T)$ is in thin position but not bridge position. The result follows from our observation on the tetrahedral triangulation and from the previous theorem.

By carefully considering regions between stable geodesics, we can improve this result, to obtain three distinct stable geodesics except in two cases. We accomplish the needed details for this in the next section.

6. THREE GEODESICS REVISITED

We begin with a theorem giving a precise description of the region between adjacent minima in a triangulation in thin position. The result yielding three stable geodesics (in most cases) appears as a corollary.

16. Definition. Let D be a triangulated disk, and let Γ_T denote the dual graph to the triangulation. A triangulated disk D is a *wheel* if Γ_T is a cycle. A triangulated disk D is a *planar lollipop* if Γ_T is isotopic to a cycle with an antenna attached. A triangulated disk D is a *fan* if Γ_T is isotopic to an arc, and the two triangles corresponding to the

endpoints of the arc share a vertex, the *distinguished vertex*, in T (see Figure 6).

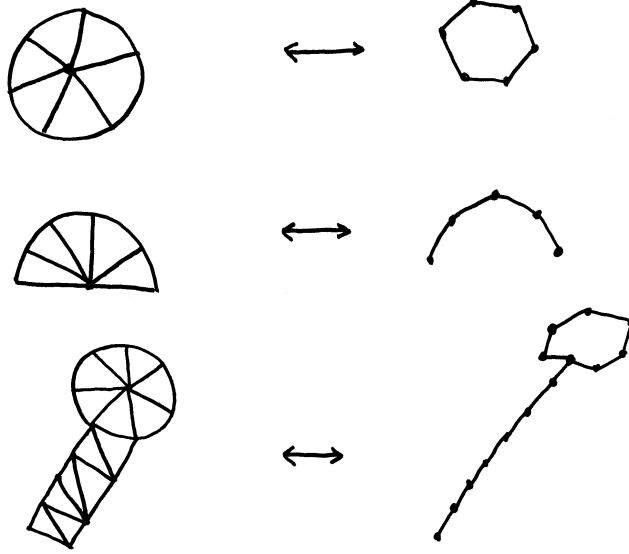


FIGURE 6. wheel, fan, planar lollipop, and their dual graphs

17. Theorem. *Let T be a triangulation of the 2-sphere. Assume T with order $O(T)$ is in thin position. Assume every cycle of length three bounds a triangle. Suppose ∂I_i and ∂I_k are stable geodesics corresponding to adjacent minima in T . Then ∂I_i and ∂I_k (or ∂I_i alone) define a subdisk G of the 2-sphere, with induced triangulation which is a wheel, a fan or a planar lollipop. If ∂I_i is adjacent to the empty geodesic, G is a wheel.*

Proof

We begin with the following Claim:

18. Claim. *Let D be a triangulated disk with a (good) ordering $O(T)$ T_1, T_2, \dots, T_m such that the ordered list $(|\partial I_1|, |\partial I_2|, \dots, |\partial I_m|)$ has a single local maximum at $|\partial I_k|, k \neq m$. If $O(T)$ is thin, then D is a wheel.*

Proof

Let $\alpha = \partial I_k$; we know that α is an unstable geodesic. Let Γ'_T be the dual graph of I_k . Γ'_T is a tree by Claim 7. The disk I_{k+1} is obtained from I_k by adding the single disk I_{k+1} . The effect of this addition on Γ'_T

is to add a single, 2-valent vertex, changing Γ'_T from a tree to a graph Γ''_T with a single cycle. If Γ''_T has any 1-valent vertices, these correspond to shortening moves for α which are disjoint from T_{k+1} . Hence the tree Γ'_T can have at most two leaves, hence it must have exactly two leaves, both of which are connected to the new vertex corresponding to T_{k+1} in Γ''_T . Hence I_{k+1} is a wheel. If I_{k+1} is a wheel with exactly three spokes, then no additional shortening move of the boundary is possible without violating the simplicial structure, so $m = k + 1$, and D is a wheel, as required. Suppose I_{k+1} is a wheel with strictly more than three spokes, and suppose $(k + 1) < m$. Let Γ'''_T be the dual graph of I_{k+2} . Γ'''_T retracts onto a theta curve, with one loop of the theta curve a cycle of length at least 4, corresponding to the wheel I_{k+1} , and one loop a cycle of length 3. We can re-order the triangles in I_{k+2} to complete the length 3 cycle first, reducing the width of the triangulation in I_{k+2} and hence in D , contradicting thinness of O . Hence if I_{k+1} is a wheel with strictly more than three spokes then $(k + 1) = m$ and $D = I_{k+1}$, which is a wheel as required.

We continue with the proof of the Theorem:

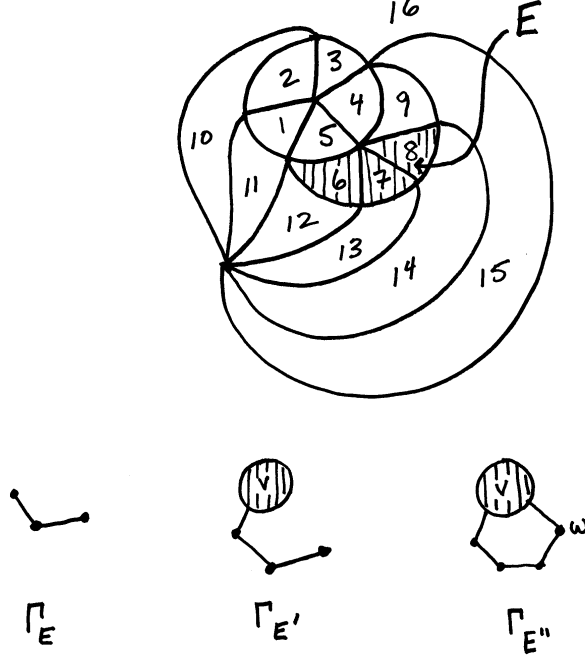
Now assume ∂I_i and ∂I_k are adjacent stable geodesics. Let $\alpha = \partial I_j$ be the unstable geodesic corresponding to the maximum that lies between $|\partial I_i|$ and $|\partial I_k|$. Let E consist of the triangles in T between $|\partial I_i|$ and $|\partial I_j|$, i.e., $E = (T_{i+1} \cup T_{i+2} \cup \dots \cup T_j)$. Let Γ_E be the dual of E . Let Γ'_E be constructed from Γ_E by adding a vertex v corresponding to the disk I_i and an edge for each triangle in E sharing an edge with ∂I_i (see Figure 7).

Note that since each triangle in $\{T_{i+1}, T_{i+2}, \dots, T_j\}$ increases the length of the boundary of I_i while leaving the homeomorphism type unchanged, Γ'_E is a tree.

Adding the triangle T_{j+1} to $I_i \cup (T_{i+1} \cup T_{i+2} \cup \dots \cup T_j)$ corresponds to adding a single bi-valent vertex w to Γ'_E ; call this new induced dual graph Γ''_E .

Recall α is an unstable geodesic. Adding w to Γ'_E is a shortening move on α . A leaf of Γ'_E which corresponds to a triangle in E also corresponds to a shortening move on α , hence the addition of T_{j+1} must eliminate all leaves of Γ'_E which correspond to triangles in E , else there will be disjoint shortening moves on opposite sides of α , a contradiction. Hence Γ'_E is a tree with at most two leaves corresponding to triangles in E , hence Γ''_E is the dual of a wheel, a fan or a planar lollipop, with one additional vertex v appended.

Our goal is to show that $j = i + 1$ or $j = k - 1$ (or possibly both); that is, we would like to see that either we cannot add any (boundary

FIGURE 7. $E = (T_6 \cup T_7 \cup T_8)$

reducing) further triangles to I_{j+1} without violating thinness, and so we are done, or else we arrived at α after adding only a single triangle to I_i . In the second case, we achieve the desired result by working backwards from the disk $S^2 - (I_k)$.

So assume Γ_E'' is the dual of a wheel, a fan or a planar lollipop, and assume that $j > i + 1$. Suppose also that $j < k - 1$. Then there is at least one additional triangle U not contained in I_{k+1} which lies in E , and adding that triangle to I_{k+1} must decrease the length of the boundary. We proceed by inspection to show that in every case, we can reduce the width of $O(T)$, violating thinness. We examine the case when Γ_E'' corresponds to a planar lollipop; the others are similar.

Note that since every cycle of length three bounds a triangle by assumption, U cannot be adjacent only to the “pop” section of the lollipop, so we need only consider the possibilities that is it adjacent to the “pop” and the stick, the stick alone, or the stick and the boundary of I_i . In each case we observe that the addition of U creates two

disjoint shortening moves for α on opposite sides, a contradiction to the assumption that $O(T)$ is thin (see one case in Figure 8).

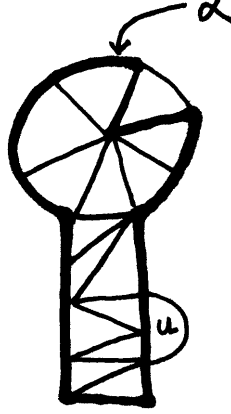


FIGURE 8. U is adjacent only to the “stick”

Since the existence of U contradicts the hypothesis that $O(T)$ is thin, we conclude that U cannot exist, hence $j = k - 1$ and E has the desired form.

19. Corollary. *Let T be a triangulation of the 2-sphere. Then either T is the tetrahedral triangulation, or the “double tetrahedral” triangulation (see Figure 9) obtained by attaching two tetrahedra along a single face, or T has at least three distinct stable geodesics.*

Proof

Assume T is not the tetrahedral triangulation. Suppose T with order $O(T)$ is in thin position. Then $O(T)$ is not bridge position. If T with $O(T)$ has at least three local minima, by Theorem 6 they each correspond to a (distinct) stable geodesic and we are done. Hence we need to consider the two cases $O(T)$ has exactly one local minimum and $O(T)$ has exactly two local minima.

Case 1: Assume $O(T)$ has exactly one local minimum. Then, by Claim 18, the unique local minimum for $O(T)$ splits the sphere into two wheels, W and V . W and V have the same number of spokes. If the number of spokes in each wheel is three, then the triangulation is the double tetrahedral triangulation and we are done.

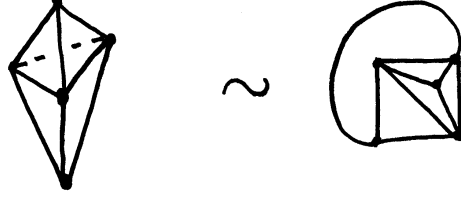


FIGURE 9. double tetrahedral triangulation

Suppose the number of spokes is at least four. Then we can find (at least) two additional stable geodesics by constructing length four paths that contain the hubs of V and W , including non-adjacent spokes in each wheel. As there will be at least two such paths (see Figure 10), and these paths are stable geodesics, the result follows.

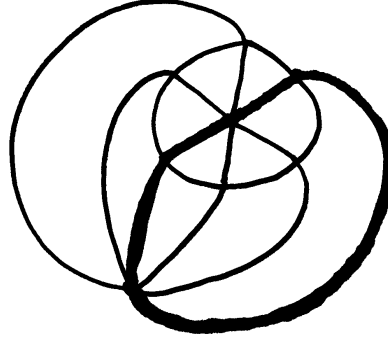


FIGURE 10. a stable geodesic through both hubs

Case 2: Assume $O(T)$ has exactly two local minima. Then the two local minima correspond to distinct stable geodesics α and β ; we need to find a third.

If every length three geodesic in T bounds a triangle, then by theorem 17, the region between the local minima, E , is a disk, and the triangulation restricted to E is a wheel, a lollipop or a fan. The triangulation in the complement of E is two wheels partly attached along

their rims. If E is a wheel or a lollipop, it contains a vertex of T in its interior, and the link of that vertex is a stable geodesic distinct from α and β . Suppose E is a fan. Let v be the distinguished vertex of E . Then E is attached to one of the complementary wheels along two edges incident to v . The link of v is again a stable geodesic, forming the boundary curve of a wheel with hub at v .

Assume there exists a length three geodesic γ in T which does not bound a triangle. Theorem 17 works as before unless γ lies in the disk E between α and β . In that case γ is distinct from α and β , and provides the third stable geodesic we are seeking.

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